Online Appendix for Theory of Decisions by Intra-Dimensional Comparisons

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Appendix: Changing the Domain

In this online appendix, we demonstrate that how to obtain the main representation result on the domain $\prod_{i=1}^{n}[a_i, b_i]$. Let $X \equiv \prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$ be the set of alternatives. Let $\succeq$ be a binary relation on X. Let $\succ$ (resp. $\sim$) denote the asymmetric (resp. symmetric) part of $\succeq$.

Now we define the model and the IDC relations. We say a function $f : \mathbb{R}^2 \to [-1, 1]$ is a *distance-based function* if it is increasing in its first argument and $f(x, y) = -f(y, x)$ for all $x, y \in \mathbb{R}$. Note that $f(x, x) = 0$ and $f(x, y) > 0$ if $x > y$. We say a function $W : [-1, 1]^n \to \mathbb{R}$ is an *aggregator* if it is increasing in all its arguments and for any i and $t_i \in [-1, 1]$, $W(t_i, \mathbf{0}_{-i}) = t_i$. Let $X_0 \equiv \{\mathbf{t} \in X | \exists i$ such that $t_i = a_i\}$. Let us call elements of X_0 as zero alternatives and elements of $X \setminus X_0$ as non zero alternatives. Similar to Theorem 1, we requires two axioms called Regularity* and Separability*. Regularity* is a collection of four postulates.

Regularity*. Let $\succeq$ be a binary relation on X.

1. (Completeness*) For any $\mathbf{x}, \mathbf{y} \in X$, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$;

2. (Continuity*) For any $\mathbf{x} \in X$, $\{\mathbf{y} \in X | \mathbf{y} \succeq \mathbf{x}\}$ and $\{\mathbf{y} \in X | \mathbf{x} \succeq \mathbf{y}\}$ are closed sets.

3. (Strong Monotonicity*) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X \setminus X_0$, if $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} > \mathbf{z}$, then $\mathbf{x} \succ \mathbf{z}$;

4. (Richness*) For any $\mathbf{x} \in X \setminus X_0$ and $\mathbf{y} \in X_0$, $\mathbf{x} \succ \mathbf{y}$.

Completeness* states that any two alternatives are comparable. Continuity requires that an upper contour set and a lower contour set of any alternative are closed. Strong monotonicity* implies the standard monotonicity axiom (for non zero alternatives): $\mathbf{x} > \mathbf{y}$ implies $\mathbf{x} \succ \mathbf{y}$. Strong monotonicity* also requires transitivity where one of three alternatives dominates one of the other two alternatives ($\mathbf{y} > \mathbf{z}$). Richness* requires that non zero alternatives are always preferred over zero alternatives.

Finally, we impose Separability* and state the representation theorem.

Separability*. For all $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in X \setminus X_0$ and i, if

$$(x_i, \mathbf{x}_{-i}) \sim (y_i, \mathbf{y}_{-i}), \quad (x_i', \mathbf{x}_{-i}) \sim (y_i', \mathbf{y}_{-i}), \text{ and}$$

$$(x_i, \mathbf{x}'_{-i}) \sim (y_i, \mathbf{y}'_{-i}), \text{ then } (x_i', \mathbf{x}'_{-i}) \sim (y_i', \mathbf{y}'_{-i}).$$

Theorem 2. A binary relation $\succeq$ on X satisfies Regularity* and Separability* if and only if there exist distance-based functions $\{f_i\}_{i=1}^n$ and an aggregator W such that for each i, f_i is continuous at $X_i^2 \setminus \{(a_i, a_i)\}$ and strictly increasing at $(a_i, b_i]^2$, $f_n(b_n, x_n) = \frac{b_n - x_n}{b_n - a_n}$ for any $x_n \in X_n = [a_n, b_n]$, W is continuous and strictly increasing at $(-1, 1)^n$, and for any $\mathbf{x} \in X \setminus X_0$ and $\mathbf{y} \in X$,

$$\mathbf{x} \succeq \mathbf{y} \text{ if and only if } W(f_1(x_1, y_1), \ldots, f_n(x_n, y_n)) \geq 0. \qquad (3)$$

Moreover, $\{f_i\}_{i=1}^n$ are unique and W is unique at $(-1, 1)^n$.

Proof of Theorem 2

Since the necessity part is obvious, we only prove the sufficiency part. Suppose a binary relation $\succeq$ satisfies Regularity* and Separability*. We then shall prove that there exist distance-based functions $\{f_i\}_{i=1}^n$ and an aggregator W such that (3) holds. First, we will prove the following useful lemma. Recall that a_k and b_k are the minimum and the maximum of $X_k = [a_k, b_k]$, respectively.

Lemma 3. Suppose $\succeq$ satisfies Regularity*. Take any $\mathbf{x} \in X \setminus X_0$. For any i and $\mathbf{y}_{-i} \in X_{-i}$ with $(b_i, \mathbf{y}_{-i}) \succeq \mathbf{x}$, there exists $y_i \in (a_i, b_i]$ such that $\mathbf{x} \sim \mathbf{y} = (y_i, \mathbf{y}_{-i})$.

Proof of Lemma 3. Take any $\mathbf{x} \in X \setminus X_0$, i, and $\mathbf{y}_{-i} \in X_{-i}$ such that $(b_i, \mathbf{y}_{-i}) \succeq \mathbf{x}$. We shall find $y_i \in (a_i, b_i]$ such that $\mathbf{x} \sim \mathbf{y}$.

By Richness*, we have $\mathbf{x} \succ (a_i, \mathbf{y}_{-i})$. We now construct two infinite sequences $\{x^n\}_{n=0}^{\infty}$ and $\{y^n\}_{n=0}^{\infty}$ by the induction. First, let us set $x^0 = b_i$ and $y^0 = a_i$. Suppose we have constructed two sequences $x^0, \ldots, x^k$ and $y^0, \ldots, y^k$. Now we will define x^{k+1} and y^{k+1} in the following way: if $(\frac{x^k+y^k}{2}, \mathbf{y}_{-i}) \succeq \mathbf{x}$, then let $x^{k+1} \equiv \frac{x^k+y^k}{2}$ and $y^{k+1} \equiv y^k$; and if $(\frac{x^k+y^k}{2}, \mathbf{y}_{-i}) \prec \mathbf{x}$, then let $x^{k+1} \equiv x^k$ and $y^{k+1} \equiv \frac{x^k+y^k}{2}$. Note that $\{x^k\}_{k=1}^{\infty}$ is a non-increasing sequence, $\{y^k\}_{k=1}^{\infty}$ is a non-decreasing sequence, and $\lim_{k\to\infty} x^k - y^k = \lim_{k\to\infty} \frac{x^0-y^0}{2^k} = 0$. So there exists $y^* \in X_i$ such that $\lim_{k\to\infty} x^k = \lim_{k\to\infty} y^k = y^* \in (a_i, b_i]$. Moreover, by the construction, we have $(x^k, \mathbf{y}_{-i}) \succeq \mathbf{x}$ and $\mathbf{x} \succ (y^k, \mathbf{y}_{-i})$ for all k. By continuity, we have $(y^*, \mathbf{y}_{-i}) \succeq \mathbf{x} \succeq (y^*, \mathbf{y}_{-i})$. Therefore, $\mathbf{x} \sim (y^*, \mathbf{y}_{-i})$. □

We use the following useful corollary of Lemma 3.

Corollary 3. For any i, j with $i \neq j$ and $x_i, y_i \in (a_i, b_i]$ with $x_i \geq y_i$, there exists $x_j \in (a_j, b_j]$ such that $(x_i, x_j, \mathbf{b}_{-i,-j}) \sim (y_i, b_j, \mathbf{b}_{-i,-j})$.

Proof of Corollary 3. Since $x_i \geq y_i$, we have $(x_i, b_j, \mathbf{b}_{-i,-j}) \succeq (y_i, b_j, \mathbf{b}_{-i,-j})$ by strong monotonicity. By Lemma 3, there exists $x_j \in (a_j, b_j]$ such that $(x_i, x_j, \mathbf{b}_{-i,-j}) \sim (y_i, b_j, \mathbf{b}_{-i,-j})$. □

Now we will prove a lemma which shows that there exist distance-based functions consistent (in some sense) with $\succeq$.

Lemma 4. If $\succeq$ satisfies Regularity* and Separability*, then there exist distance-based functions $\{f_i\}_{i=1}^n$ such that f_i is continuous at $X_i^2 \setminus \{(a_i, a_i)\}$ and strictly increasing at $(a_i, b_i]^2$, and for any $\mathbf{x}, \mathbf{y} \in X \setminus X_0$, i, and $x_i', y_i' \in (a_i, b_i]$, if $\mathbf{x} \sim \mathbf{y}$ and $f_i(x_i, y_i) = f_i(x_i', y_i')$, then $(x_i', \mathbf{x}_{-i}) \sim (y_i', \mathbf{y}_{-i})$.

Proof of Lemma 4. First, let us construct distance-based functions.

For each $i < n$, let f_i be a function such that for any $x_i, y_i \in (a_i, b_i]$ with $x_i > y_i$, $f_i(x_i, y_i) = \frac{b_n - x_n}{b_n - a_n}$ and $f_i(y_i, x_i) = -\frac{b_n - x_n}{b_n - a_n}$ whenever $(x_i, x_n, \mathbf{b}_{-i,-n}) \sim (y_i, \mathbf{b}_{-i})$. By Corollary 3, f_i is well-defined. Moreover, let $f_i(x_i, a_i) = 1$ for any $x_i > a_i$. By continuity and strong monotonicity, f_i is also continuous at $X_i^2 \setminus \{(a_i, a_i)\}$ and strictly increasing in its first argument at $(a_i, b_i]^2$.

Now, we will construct f_n. By Corollary 3, for any $x_n, y_n \in (a_n, b_n]$ with $x_n > y_n$, there exists $x_1 \in (a_1, b_1]$ such that $(x_1, x_n, \mathbf{b}_{-1,-n}) \sim (y_n, \mathbf{b}_{-n})$. Let f_n be a function such that for any $x_n, y_n \in (a_n, b_n]$ with $x_n > y_n$, $f_n(x_n, y_n) = f_1(b_1, x_1)$ and $f_n(y_n, x_n) = f_1(x_1, b_1)$ whenever $(x_1, x_n, \mathbf{b}_{-1,-n}) \sim (y_n, \mathbf{b}_{-n})$. Moreover, let $f_n(x_n, a_n) = 1$ for any $x_n > a_n$. Similarly, f_n is well-defined, continuous at $X_n^2 \setminus \{(a_n, a_n)\}$, and strictly increasing in its first argument at $(a_n, b_n]^2$. Also, note that by the definition of f_1, $f_n(b_n, x_n) = \frac{b_n - x_n}{b_n - a_n}$ for all $x_n \in X_n$.

Now, we will prove that we constructed desired distance-based functions. Take any $\mathbf{x}, \mathbf{y} \in X$, i, and $x_i', y_i' \in X_i$ such that $\mathbf{x} \sim \mathbf{y}$ and $f_i(x_i, y_i) = f_i(x_i', y_i')$. We shall prove that $(x_i', \mathbf{x}_{-i}) \sim (y_i', \mathbf{y}_{-i})$. Without loss of generality, suppose $x_i \geq y_i$. We consider two cases.

Case 1: $i < n$.

Take some $\bar{x}_n \in (a_n, b_n]$ such that $f_i(x_i, y_i) = f_i(x_i', y_i') = \frac{b_n - \bar{x}_n}{b_n - a_n}$. By the definition of f_i, we obtain $(x_i, \bar{x}_n, \mathbf{b}_{-i,-n}) \sim (y_i, \mathbf{b}_{-i})$ and $(x_i', \bar{x}_n, \mathbf{b}_{-i,-n}) \sim (y_i', \mathbf{b}_{-i})$. Since $\mathbf{x} \sim \mathbf{y}$, $(x_i, \bar{x}_n, \mathbf{b}_{-i,-n}) \sim (y_i, \mathbf{b}_{-i})$, and $(x_i', \bar{x}_n, \mathbf{b}_{-i,-n}) \sim (y_i', \mathbf{b}_{-i})$, by Separability*, $(x_i', \mathbf{x}_{-i}) \sim (y_i', \mathbf{y}_{-i})$ holds.

Case 2: $i = n$.

Take some $\bar{x}_1 \in (a_1, b_1]$ such that $f_n(x_n, y_n) = f_n(x_n', y_n') = f_1(b_1, \bar{x}_1)$. By the definition of f_n, we obtain $(\bar{x}_1, x_n, \mathbf{b}_{-1,-n}) \sim (y_n, \mathbf{b}_{-n})$ and $(\bar{x}_1, x_n', \mathbf{b}_{-1,-n}) \sim (y_n', b_{-n})$. Since $\mathbf{x} \sim \mathbf{y}$, $(\bar{x}_1, x_n, \mathbf{b}_{-1,-n}) \sim (y_n, \mathbf{b}_{-n})$, and $(\bar{x}_1, x_n', \mathbf{b}_{-1,-n}) \sim (y_n', \mathbf{b}_{-n})$, by Separability*, $(x_n', \mathbf{x}_{-n}) \sim (y_n', \mathbf{y}_{-n})$ holds. □

Corollary 4. For any i, the range $f_i(X_i^2) = [-1, 1]$.

Proof of Corollary 4. When $i = n$, note that $f_i(b_n, x_n) = \frac{b_n - x_n}{b_n - a_n}$ can take any number in $[0, 1]$ by appropriately choosing x_n. Therefore, $f_n(X_n^2) = [-1, 1]$. Now suppose $i < n$. By Corollary 3, for any $x_n \in (a_n, b_n]$, there exists $x_i \in (a_i, b_i]$ such that $(x_i, b_n, \mathbf{b}_{-i,-n}) \sim (b_i, x_n, \mathbf{b}_{-i,-n})$. By the construction of f_i, we have $f_i(b_i, x_i) = f_n(b_n, x_n) = \frac{b_n - x_n}{b_n - a_n}$. Therefore, $(-1, 1) \subset f_i(X_i^2)$. Moreover, since $f_i(x_i, a_i) = 1$ for any $x_i > a_i$, we have $f_i(X_i^2) = [-1, 1]$. □

Finally, we will prove Theorem 2. We shall find increasing function W such that W is continuous and strictly increasing at $(-1, 1)^n$ and $W(t_i, \mathbf{0}_{-i}) = t_i$ and $W(\mathbf{t}) = W(\mathbf{t}_{-n}, 0_n) + t_n$ for any $t \in (-1, 1)^n$. We construct W in the following way. First, for any $\mathbf{t}_{-n} \in (-1, 1)^{n-1}$, we construct $W(\mathbf{t}_{-n}, 0_n)$. Then for any $\mathbf{t} \in (-1, 1)^n$, we set $W(\mathbf{t}) \equiv W(\mathbf{t}_{-n}, 0_n) + t_n$. In order to construct $W(\mathbf{t}_{-n}, 0_n)$, for each $\mathbf{t}_{-n} \in (-1, 1)^{n-1}$ we will find $\bar{x}_n, \bar{y}_n \in (a_n, b_n]$ (to be described later) and set $W(\mathbf{t}_{-n}, 0_n) \equiv f_n(\bar{y}_n, \bar{x}_n)$.

Take $\mathbf{t}_{-n} \in (-1, 1)^{n-1}$. We will construct $\bar{x}_n$ and $\bar{y}_n$ by the following two claims.

Claim 3: For any $j < n$, there exist $\bar{x}_j, \bar{y}_j \in (a_j, b_j]$ such that $t_j = f_j(\bar{x}_j, \bar{y}_j)$.

For any j, note that $f_j(b_j, (a_j, b_j]) = [0, 1)$ by the construction of f_j. Therefore, there exists $z_j \in (a_j, b_j]$ such that $f_j(b_j, z_j) = |t_j|$. Then we have $t_j = f_j(\bar{x}_j, \bar{y}_j)$ by setting $(\bar{x}_j, \bar{y}_j) = (b_j, z_j)$ when $t_j \geq 0$ and $(\bar{x}_j, \bar{y}_j) = (z_j, b_j)$ when $t_j < 0$.

Now suppose we have constructed $\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{y}_1, \ldots, \bar{y}_{n-1}$ by Claim 3.

Claim 4: There exist $\bar{x}_n, \bar{y}_n \in (a_n, b_n]$ such that $\bar{\mathbf{x}} \sim \bar{\mathbf{y}}$.

We consider two cases. First, suppose $(b_n, \bar{\mathbf{x}}_{-n}) \succeq (b_n, \bar{\mathbf{y}}_{-n})$. By Corollary 3, there exists $z_n \in (a_n, b_n]$ such that $(z_n, \bar{\mathbf{x}}_{-n}) \sim (b_n, \bar{\mathbf{y}}_{-n})$. So we set $(\bar{x}_n, \bar{y}_n) = (z_n, b_n)$. Second, suppose $(b_n, \bar{\mathbf{x}}_{-n}) \prec (b_n, \bar{\mathbf{y}}_{-n})$. By Corollary 3, there exists $z_n \in (a_n, b_n]$ such that $(b_n, \bar{\mathbf{x}}_{-n}) \sim (z_n, \bar{\mathbf{y}}_{-n})$. Similarly, $z_n \neq a_n$. So we set $(\bar{x}_n, \bar{y}_n) = (b_n, z_n)$.

Now let $W(\mathbf{t}_{-n}, 0_n) \equiv f_n(\bar{y}_n, \bar{x}_n)$ and $W(\mathbf{t}) \equiv W(\mathbf{t}_{-n}, 0_n) + t_n$ for any $\mathbf{t} \in (-1, 1)^n$. By this construction, W is continuous and strictly increasing in all its arguments at $(-1, 1)^n$. Finally, we extend W to $[-1, 1]^n$. Take any $\mathbf{t} \in [-1, 1]^n \setminus (-1, 1)^n$. First, if there are i and j such that $t_i = 1$ and $t_j = -1$, then we set $W(\mathbf{t}) = 0$. Second, if there is i such that $t_i = 1$, but no j such that $t_j = -1$, then we set $W(\mathbf{t}) = 1$. Finally, if there is i such that $t_i = -1$, but no j such that $t_j = 1$, then we set $W(\mathbf{t}) = -1$.

Now we shall prove that for any $\mathbf{x} \in X \setminus X_0$ and $\mathbf{y} \in X$,

$$\mathbf{x} \succeq \mathbf{y} \text{ if and only if } W\big((f_i(x_i, y_i))_i\big) \geq 0.$$

When $\mathbf{y} \in X_0$, then by Richness*, $\mathbf{x} \succ \mathbf{y}$. Note that for each i, $f_i(x_i, y_i) > -1$ since $x_i \neq a_i$. Moreover, for each j such that $y_j = a_j$, we have $f_i(x_i, y_i) = 1$. Therefore, by the construction of W, $W\big((f_i(x_i, y_i))_i\big) = 1 > 0$. Now we consider the case $\mathbf{x}, \mathbf{y} \in X \setminus X_0$. Consequently, $(f_i(x_i, y_i))_{i=1}^n \in (-1, 1)^n$.

Since W is continuous and strictly increasing in all its arguments at $(-1, 1)$ and each f_i is continuous and strictly increasing in its first argument at $(a_i, b_i]^2$, it is enough to prove that $\mathbf{x} \sim \mathbf{y}$ implies $W\big((f_i(x_i, y_i))_i\big) = 0$.

Take any $\mathbf{x}, \mathbf{y} \in X \setminus X_0$ with $\mathbf{x} \sim \mathbf{y}$. First, let $\mathbf{t}_{-n} = ((f_i(x_i, y_i))_{i<n})$. Then by the above procedure that involves Claims 3-4, we find $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in X \setminus X_0$ such that $f_j(\bar{x}_j, \bar{y}_j) = t_j$ for all $j < n$ and $\bar{\mathbf{x}} \sim \bar{\mathbf{y}}$. By the construction of W, we have $W\big((f_i(x_i, y_i))_i\big) = W\big((f_i(x_i, y_i))_{i<n}, 0_n\big) + f_n(x_n, y_n) = f_n(\bar{y}_n, \bar{x}_n) + f_n(x_n, y_n)$.

Since $f_1(x_1, y_1) = f_1(\bar{x}_1, \bar{y}_1) = t_1$, by Lemma 4, $\mathbf{x} \sim \mathbf{y}$ implies $(\bar{x}_1, \mathbf{x}_{-1}) \sim (\bar{y}_1, \mathbf{y}_{-1})$. Similarly, since $f_2(x_2, y_2) = f_2(\bar{x}_2, \bar{y}_2) = t_2$, by Lemma 4, $(\bar{x}_1, \mathbf{x}_{-1}) \sim (\bar{y}_1, \mathbf{y}_{-1})$ implies $(\bar{x}_1, \bar{x}_2, \mathbf{x}_{-1,-2}) \sim (\bar{y}_1, \bar{y}_2, \mathbf{y}_{-1,-2})$. Since $f_j(\bar{x}_j, \bar{y}_j) = t_j$ for all $j < n$, by repeating this argument $n-1$ times, we will obtain that $(\bar{\mathbf{x}}_{-n}, x_n) \sim (\bar{\mathbf{y}}_{-n}, y_n)$. Moreover, since $\bar{\mathbf{x}} \sim \bar{\mathbf{y}}$, by Lemma 4, $(\bar{\mathbf{x}}_{-n}, x_n) \sim (\bar{\mathbf{y}}_{-n}, y_n)$ implies $f_n(y_n, x_n) = f_n(\bar{y}_n, \bar{x}_n)$. In other words, $W\big((f_i(x_i, y_i))_i\big) = f_n(\bar{y}_n, \bar{x}_n) + f_n(x_n, y_n) = 0$.

Lastly, we show that W is an aggregator; that is, $W(t_i, \mathbf{0}_{-i}) = t_i$ for any $t_i \in (-1, 1)$. By the construction of W, it is obvious when $i = n$. Now take any $i < n$ and $t_i \in [0, 1)$. There exist $x_i, y_i \in (a_i, b_i]$ such that $f_i(x_i, y_i) = -t_i = f_n(b_n(1 - t_i) + a_n t_i, b_n)$. By the construction of f_i, we have $(x_i, \mathbf{b}_{-i}) \sim (y_i, b_n(1 - t_i) + a_n t_i, \mathbf{b}_{-i,-n})$. Then since $W(\mathbf{t}) = W(\mathbf{t}_n, 0_n) + t_n$ for each $\mathbf{t} \in (-1, 1)^n$, we obtain $0 = W\big(f_i(y_i, x_i), f_n(b_n(1 - t_i) + a_n t_i, b_n), \mathbf{0}_{-i,-n}\big) = W\big(f_i(y_i, x_i), \mathbf{0}_{-i}\big) +$

$f_n(b_n(1-t_i)+a_nt_i, b_n) = W\big(t_i, \mathbf{0}_{-i}\big) - t_i$. A similar argument works for any $t_i \in (-1,0]$.

Uniqueness: Suppose there are two sets of functions $(W, \{f_i\}_{i=1}^n)$ and $(W', \{f_i'\}_{i=1}^n)$ such that $W(\mathbf{t}) = W(\mathbf{t}_{-n}, 0_n) + t_n$ and $W'(\mathbf{t}) = W'(\mathbf{t}_{-n}, 0_n) + t_n$ for any $\mathbf{t} \in (-1,1)^n$ and $f_n(b_n, x_n) = f_n'(b_n, x_n) = \frac{b_n - x_n}{b_n - a_n}$ for any $x_n \in X_n$ that satisfy (3). We shall prove that $f_i = f_i'$ for any i and $W = W'$ at $(-1,1)^n$.

Take any $i < n$ and $x_i, y_i \in X_i$ with $x_i > y_i$. We shall prove that $f_i(x_i, y_i) = f_i'(x_i, y_i)$. When $y_i = a_i$, we have $f_i(x_i, y_i) = f_i'(x_i, y_i) = 1$. Now suppose $y_i > a_i$. By Corollary 3, there exists $x_n \in (a_n, b_n]$ such that $(x_i, x_n, \mathbf{b}_{-i,-n}) \sim (y_i, \mathbf{b}_{-i})$. By (3), we have

$$W\big(f_i(x_i,y_i), \mathbf{0}_{-i,-n}, f_n(x_n,b_n)\big) = W\big(f_i(x_i,y_i), \mathbf{0}_{-i}\big) + f_n(x_n,b_n) = f_i(x_i,y_i) - \frac{b_n - x_n}{b_n - a_n} = 0.$$

Similarly,

$$W'\big(f_i'(x_i,y_i), \mathbf{0}_{-i,-n}, f_n'(x_n,b_n)\big) = f_i'(x_i,y_i) - \frac{b_n - x_n}{b_n - a_n} = 0.$$

Then we will obtain that $f_i(x_i,y_i) = f_i'(x_i,y_i) = \frac{b_n - x_n}{b_n - a_n}$. Therefore, $f_i = f_i'$ for any $i < n$.

Now we will prove that $f_n = f_n'$. Take any $x_n, y_n \in X_n$ with $x_n < y_n$. We shall prove that $f_n(y_n, x_n) = f_n'(y_n, x_n)$. When $x_n = a_n$, we have $f_n(y_n, x_n) = f_n'(y_n, x_n) = 1$. Now suppose $x_n > a_n$. By Corollary 3, there exists $y_1 \in X_1$ such that $(x_n, \mathbf{b}_{-n}) \sim (y_1, y_n, \mathbf{b}_{-1,-n})$. By (3),

$$\begin{aligned} 0 = W\big(f_1(b_1,y_1), \mathbf{0}_{-1,-n}, f_n(x_n,y_n)\big) &= W\big(f_1(b_1,y_1), \mathbf{0}_{-1}\big) + f_n(x_n,y_n), \text{ (since } W(\mathbf{t}) = W(\mathbf{t}_{-n}, 0_n) + t_n) \\ &= f_1(b_1,y_1) + f_n(x_n,y_n), \text{ (by the definition of aggregator),} \end{aligned}$$

and

$$W'\big(f_1'(b_1,y_1), \mathbf{0}_{-1,-n}, f_n'(x_n,y_n)\big) = f_1'(b_n,y_1) + f_n'(x_n,y_n) = 0.$$

Since $f_1 = f_1'$, we obtain that $f_n(y_n,x_n) = f_n'(y_n,x_n) = f_1(b_1,y_1)$. Therefore, $f_n = f_n'$.

Finally, we shall prove that $W = W'$ at $(-1,1)^n$. Since $W(\mathbf{t}) = W(\mathbf{t}_{-n}, 0_n) + t_n$ and $W'(\mathbf{t}) = W'(\mathbf{t}_{-n}, 0_n) + t_n$, it is enough to prove that $W(\mathbf{t}_{-n}, 0_n) = W'(\mathbf{t}_{-n}, 0_n)$ for any $\mathbf{t}_{-n} \in (-1,1)^{n-1}$.

Now take any $\mathbf{t}_{-n} \in (-1,1)^{n-1}$. For each $i < n$, there exist $x_i, y_i \in X_i$ such that $t_i = f_i(x_i,y_i) = f_i'(x_i,y_i)$ by Claim 3. Moreover, by Claim 4, there exist $x_n, y_n \in X_n$ such that

$\mathbf{x}\sim\mathbf{y}$. Let $t_n = f_n(x_n, y_n) = f'_n(x_n, y_n)$. Therefore,

$$
\begin{aligned}
W(\mathbf{t}_{-n}, 0) &= W(\mathbf{t}_{-\mathbf{n}}, t_n) - t_n = 0 - t_n, \text{ (by } \mathbf{x}\sim\mathbf{y} \text{ and (3))}, \\
&= W'(\mathbf{t}_{-\mathbf{n}}, t_n) - t_n, \text{ (by } \mathbf{x}\sim\mathbf{y} \text{ and (3))}, \\
&= W'(\mathbf{t}_{-\mathbf{n}}, 0).
\end{aligned}
$$

Therefore, $W = W'$ at $(-1, 1)^n$.